TEACHING MATHEMATICAL PROBLEM SOLVING

Using MOEMS® Contest Problems

Jason Frand with Ruth Sabean

Edited by

Nicholas J Restivo and Emma Sabean

Why Teach Mathematical Problem Solving Using MOEMS® Contest Problems?

There is a simple answer to this question. Problems from the contests of Mathematical Olympiads for Elementary & Middle Schools (MOEMS®) will engage you and your students to think, speak, and understand mathematical concepts in a way that your curriculum might lack. They will become involved in conversations about math: different ways of approaching problems, sharing their ideas, arguing for the "best" way to do a given problem, and oftentimes berating themselves by saying, "I should have gotten that one!"

In the past few years, more and more parents and interested individuals have been asking to be PICOs (**P**erson **I**n **C**harge of the **O**lympiads), but have asked how best to prepare their kids for the contests. For a long while, Jason Frand and Ruth Sabean have been sharing their excitement over their methods of getting kids ready to take our contests, but also to make them better problem solvers in all endeavors. Together, we came up with the idea of revealing those tried-and-true methods with you. At last, we have put together a valuable classroom resource to help meet that need. Get ready to "Excite, Energize, and Enrich" your classroom using MOEMS problems.

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Preface: Why Write a Book of Instructional Guides?

A friend said to us "Oh, so you teach math?" and we answered "No, we teach how to use math." What's the difference?

Consider the differences between *teaching arithmetic* and *teaching how to use arithmetic*. Learning arithmetic—basic addition, subtraction, multiplication and division—is critical for every child. There are many approaches to how best to teach these concepts. Over the years, we have seen success using Singapore Math and Cognitive Guided Instruction, and there are numerous other successful approaches. We do not address any of those ideas in this book. Learning to count, add, and divide, as well as to use fractions, positive and negative numbers, and so many other topics are outside what we will address. Our challenge is to teach students how to *use* what they already know, along with their innate logical reasoning skills, to solve unfamiliar looking problems.

Teaching basic mathematical ideas and teaching mathematical problem solving are equally challenging. They require different approaches, assumptions and preparation. In this book, we present the successful methods we have developed for teaching mathematical problem solving. This book provides both detailed instructional plans for individual problems that can assist a teacher who is preparing to teach mathematical problem solving and a general method of breaking down new problems.

Problem solving has been recognized as a critical core value of mathematics education at all grade levels. This is reflected in the wide spread adoption of the Common Core State Standards for Mathematics. Students are expected to not only complete the work but also explain their work. The ultimate goal is for each student to become an independent problem solver. The teachers' role is to help each student achieve this goal. *Teaching Mathematical Problem Solving Using MOEMS Contest Problems* is designed to help elementary and middle school teachers acquire the knowledge they need to confidently teach students how to understand math word problems and persevere in solving them.

The MOEMS Contest Problems are very challenging for many students. Even though they may get low scores on the competitions and struggle with the problems, students who love to do math just keep asking to do more problems!

Why?

We think it is the "Ah ha!" Those eureka moments: "I got it!" After a student struggles for some time with a problem, there comes a sense of joy, satisfaction, and understanding. That's the moment which provides a sense of accomplishment and self-pride that makes the work worthwhile. They know the problem was hard, they may even know they didn't solve it correctly, but when explained, the sense of "Yes!!! I understand." is so powerful, that students keep coming back for more! For many students, those moments are like magic.

How do we create opportunities for a magical learning experience?

Traditionally, mathematicians use a four-step approach to solve a problem: understand the problem, figure out a strategy, execute that strategy, and check your work. This is the method we try to teach to our students, with special emphasis on the importance of reflecting on a problem before engaging in solving it. However, it does not fit how most students actually go about problem solving, namely, just picking up their pencils and starting to write. In response, we developed a slogan, which we call the KEY:

Stop! Think…Go…Think…Go…Think…Go…

The KEY reminds students to think before they "go" and to also keep doubling back and thinking about the problem some more, to ensure that their work is relevant and captures all the nuances of the problem.

Our understanding has grown over the years as we observed how students learn and solve problems. As a result, we developed two very powerful instructional instruments:

1. A Problem Assessment tool to guide teachers in planning a lesson.
2. A Solution Strategy Worksheet to engage the students in analytical thinking.

These instructional instruments have changed the way we teach in very positive ways and are discussed in detail in this book.

Changing how to teach is very difficult. In our early years of teaching the MOEMS Contest Problems, we taught the way we had been taught as students. We gave our students many problems and either called on the person with the correct answer or put the correct answer on the board ourselves. We asked specific questions to solicit the specific answers we wanted. In retrospect, we made our classroom a very teacher-centered environment.

Our classroom is now a problem-oriented learning environment. We may have students tackle only one or two problems in a lesson. Working in small groups, students engage in guided mathematical discussions around possible strategies to solve a problem. They are learning to read for understanding. They are developing analytical skills as well as broadening their exposure to a variety of solution strategies through experimentation, observation, discussion and evaluation. We are moving around the room, observing our students' work and asking questions to help move them toward a broader understanding of the complex problems in which they are engaged.

Our classroom is noisy with the excitement of learning how to solve problems using mathematics!

We wrote this book to provide practical classroom-ready information for teachers who want to foster a problem-oriented learning environment. In an ideal world, teachers would be able to collaborate, as we have done, to develop lessons that anticipate student needs and include questions to advance student thinking. Our goal is to collaborate with teachers who use this book by providing detailed Instructional Guides that make collaboration possible!

Upon retiring from UCLA, Jason and Ruth team-taught a math enrichment class at a Title 1 school. The MOEMS program of problems and contests was selected to provide structure around which instruction could be based.

Over the past decade, we discussed how to improve our teaching of the problems and prepare our students for the monthly MOEMS contests. It was through these experiences as well as the teaching of professional development classes on mathematical problem solving that this book emerged.

We feel very privileged to be part of this critically important emphasis on problem solving.

—Jason and Ruth

Chapter 1: What is an Instructional Guide

As articulated in the Common Core Mathematics Standards, problem solving is recognized as a critical core value of mathematics education at all grade levels. Students are expected to not only complete the work, but also to be able to explain their work. The ultimate goal is for each student to be an independent problem solver. The teachers' role is to help each student achieve these goals. This book is designed to help elementary and middle school teachers acquire the knowledge they need to confidently achieve the objectives of helping students make sense of math word problems and persevere in solving them.

This book is organized to provide teachers with the following information:

- How to introduce a math word problem,
- How to help students learn to unpack[2] math word problems,
- How to expose students to a diversity of strategies and tools for solving math word problems,
- How to engage students in mathematical thinking to better guide them in solving math word problems,
- How to create a curriculum based on MOEMS Contest Problems.

Twenty-eight MOEMS Contest Problems are thoroughly analyzed and presented to provide teachers with a specific "how to" list. These problems were selected from over 2000 published MOEMS Contest Problems[3] specifically to reflect the most common strategies and tools needed by students to successfully solve the problems. The selected problems along with instructional guides will provide teachers with sufficient breadth and depth of knowledge to confidently teach any MOEMS Contest Problem.

An **Instructional Guide** is used to present the MOEMS Contest Problems. To help us determine what should be included in the Instructional Guides, we showed options to several teachers and asked which would be the most useful. The information and layout in this book reflects their feedback.

Each Instructional Guide is for *one* problem and includes:

- **A Problem Assessment** to identify potential roadblocks students may encounter,

[2] "Unpacking a problem" is a term used in many math curricula to describe the process of making sense of a problem. See page 54 for details.

[3] Each year, in November, December, January, February and March, MOEMS distributes a competition electronically. Each competition has five problems and is administered by individual teachers to their classes. The published volumes are collections of past MOEMS Contest Problems. See moems.org for information.

- **Alternative strategies** for solving the problem,
- **Correct solutions** for solving the problem,
- **Partial solutions** with questions to guide students as they engage in solving a problem,
- **Failed starts** with questions to help students understand and start the problem,
- **A place to record notes** for the next time the problem is taught,
- **A Solution Strategy Worksheet** for students that provides structured problem analysis to extend their thinking,
- **References** to guide teachers to other similar problems for follow-up instruction.

All of these items will be discussed in detail in Chapter 13. For a brief overview, please turn to page 94 to look at the Instructional Guide for Problem #1.

Solution Strategy

In this book, the term **solution strategy** is used to refer to the overall approach used to answer the questions: "Where do I start the problem? What do I need to do to solve the problem?"

Appendix 1 provides definitions for the eight solution strategies most commonly occurring in the Instructional Guides. There are many more strategies described elsewhere such as inductive reasoning, list reasoning, or simplification reasoning.

Strategies and Tools: What is the difference?

In some math curricula, the terms strategy and tool are used for two distinct concepts. In all MOEMS volumes, as in this book, the terms will be used interchangeably. For example, when referring to pencil and paper, manipulating blocks or counters, making a diagram or table, using a calculator or computer, creating an organized list or working backward, no distinction will be made. Historically, mathematicians have not made a distinction and we will follow that convention in this book.

Student Work as Anchor Pages

A common concern we heard from teachers when planning this book was "we don't understand the hints and solutions in the MOEMS's books." Another was "I don't have enough math knowledge to understand my students' errors." To address these and similar concerns, the Instructional Guides use samples of student work as the basis of the mathematical discussions so that material is grade appropriate.

The samples of student solutions are called **anchor pages**. For each problem, there is at least one correct anchor page, one partial solution anchor page and one failed start anchor page. When more than one anchor page is included for a solution category, each anchor page shows a different approach to solving the problem. Each anchor page has accompanying analysis and sample questions that teachers may ask to clarify or extend the students' thinking.

From Student to Mathlete

From day one in our classes, we emphasized that MOEMS problems are about learning to solve extremely difficult and challenging problems and that it is very rare that any student gets them all correct. We tell our students that these problems are designed to strengthen their thinking skills. It is the work that counts—learning to deal with complex and challenging problems is fun! It is the participation that makes it all worthwhile. We call our students **"mathletes"** because they are given the opportunity and additional training to be much more than just "good in math." We tell them that just as with any activity in which they would like to improve, be it soccer or piano, practice is critical to becoming successful.

From this point forward, we will use the term mathlete when referring to students who are learning how to solve MOEMS problems.

The Instructional Guide in Action

The primary purpose of the Instructional Guides is to provide teachers with an understanding of the problems, an understanding of how mathletes try to solve the problems, and options for how they can guide mathletes to understand and solve problems. To achieve these purposes, specific examples of mathlete thinking are shown as anchor pages along with an analysis and Sample Questions to Extend Mathlete Learning.

This book has Instructional Guides for 28 problems, a sufficient number, variety, and level of difficulty to provide teachers with related mathematical ideas for a range of problem types. Our goal is to help prepare teachers to use any MOEMS contest problem and know they have an approach to teaching the problem. There are many pedagogical and curricular questions related to the use of the MOEMS contest problems. These are discussed in Chapter 6.

A Problem Assessment, which identifies potential roadblocks, is provided for each problem. The anchor pages provide examples of mathlete thinking and illustrate the roadblocks. The analysis and questions included with each anchor page are intended to clarify the roadblocks. Our goal is for teachers to use this material as they prepare a lesson for introducing the problem and guiding mathletes to understand and solve the problem.

Discussing Instructional Guides Online

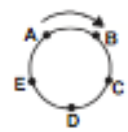
We have designed the Instructional Guides to be as useful a teaching aid as possible. The book binding was selected specifically to enable ease of photo copying of worksheets as well as easy access to Instructional Guides. Space is also provided to record a specific teaching experience. We encourage instructors to add notes about teaching experiences, which can be reviewed the next time a problem is taught. In addition, MOEMS has set up a web discussion group to support communication among instructors about the process of teaching MOEMS problems. Please log in to your Facebook account and search for...

"TMPS-Teaching Mathematical Problem Solving-Discussion."

Instructional Guide for Problem #9

MOEMS Success Rate: 84%
MOEMS Problem Type: Circles, Cycling numbers
Solution Strategy: Diagram reasoning
MOEMS Problem Number: 3-2-2A

Problem #9: An ant travels around the circle in the direction shown. It touches each of the labeled points in order. The first three points that the ant touches are A, B, and C, in that order. What is the 28th point that the ant touches?



MOEMS Solution

METHOD 1: *Strategy: Count by complete circuits of the circle.*
The letter E is touched every 5 points beginning with the fifth point. Thus E is the 25th touch and **C is the 28th point the ant touches.**

METHOD 2: *Strategy: Count by individual points.*
The points in order are **ABCDE ABCDE …** . The 28th point that the ant touches is C.

Problem Assessment

Filled out for problem taught in September.

Attribute	5th-Grade Rating	Potential Roadblocks
Math vocabulary	Known	
Math concepts	Known	
Wording	Simple	
Translation	Simple	
Solution strategy	Known	

Note: Because all five attributes are rated "known" or "simple," no potential roadblocks are listed for this problem.

Problem Assessment

for problem to be taught on ______________________.

Attribute	Your Class Rating		Potential Roadblocks
Math vocabulary	Known	New	
Math concepts	Known	New	
Wording	Simple	Complex	
Translation	Simple	Complex	
Solution strategy	Known	New	

Plans to address roadblocks:

Notes for next time

I taught the lesson on __________________.

1. Overall, I felt the lesson went…

2. I adjusted the Problem Assessment as…

3. Potential roadblocks I need to consider next time…

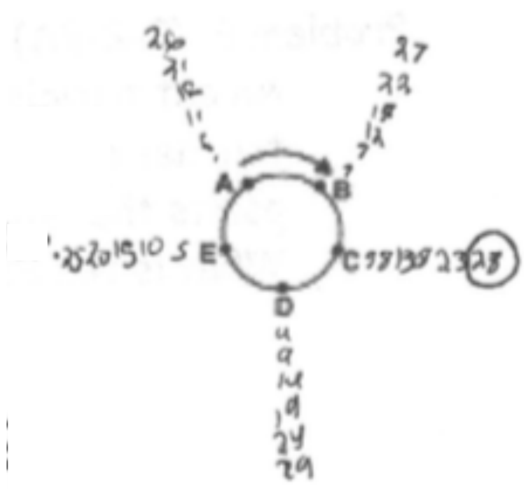
4. Other questions I could ask…

5. I had the students work (circle all that apply and comment)
   4) Alone
   5) With a learning partner
   6) In a small group

6. Next time I teach this problem…

Correct Solution Strategies

A:

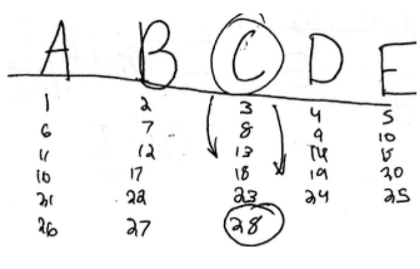


This "pinwheel" approach was a common strategy. It provided a straightforward counting solution to the problem. Those who counted correctly found that C is the 28th point the ant touches.

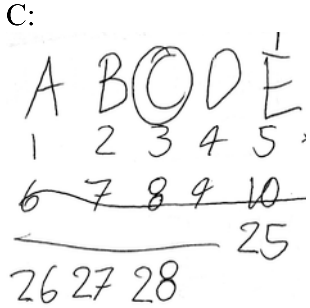
The solution strategies of Mathletes B, C and D show an evolution of mathematical growth. All three mathletes listed the five points and then proceeded to count to 28.

Mathlete B listed each number; **Mathlete C** counted by fives; **Mathlete D** did the division and used only the remainder to count to letter C.

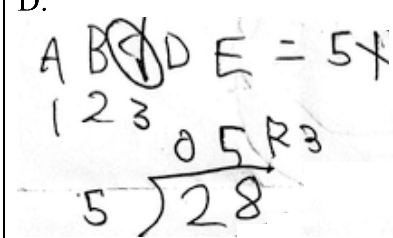
B:



C:



D:

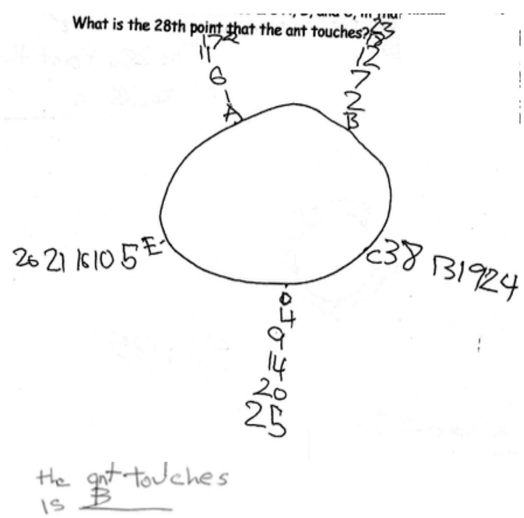
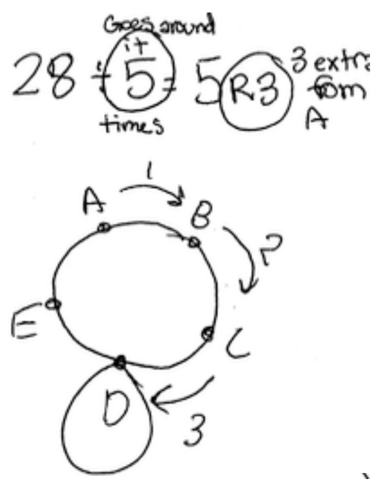


Sample Questions to Extend Mathlete Learning

- Describe your thinking in solving the problem.
- If we wanted to know the 68th point touched is there an efficient way to figure it out?
- Did you try another way to solve the problem?
- How did you check your work?

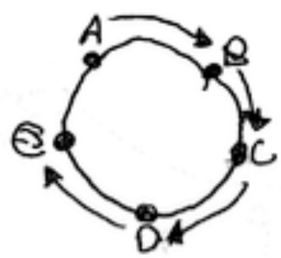
Notes for next time:

Partial Solution Strategies

E: 	**Mathlete E** has written an open sentence indicating they know what they are trying to find. However, the mathlete made counting errors (for example, skipping 15), revealing a drawback in the solution strategy of writing every number. **Sample Questions to Get Unstuck** • Describe your thinking in solving the problem. • Look at the numbers at point A. Do you see a pattern? • Look at the numbers at point B. Do you see a pattern? • What should the pattern of numbers be at each point? • What is the pattern at point C? • How could the mathlete have used the patterns to check their work?
F: 	**Mathlete F**'s analysis is excellent and essentially the same as **Mathlete D**. However, by numbering the journey the ant travels instead of the points touched, this mathlete counted "extra from A" rather than counting A as the first point when counting out the remainder of 3. **Sample Questions to Get Unstuck** • Describe your work. • Explain 28/5 = 5 R3. • On what letter is the ant standing after one times around the circuit? After 5 times? • What are the last three points the ant will touch?

Notes for next time:

Failed Start

G: 	This was the total extent of work on this mathlete's page. It appears **Mathlete G** understood that the ant would move from A to B to C and so on, but it seems they thought it stopped at E, or perhaps they ran out of time.

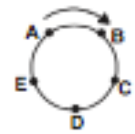
Sample Questions to Lead to Understanding

- What do we need to find? Write an open sentence.
- How is the ant traveling?
- What does "clockwise" mean?
- What is the label of the starting point?
- How many letters are there in the circle? What are there the letters?
- What is the last letter in each circle?
- How many points does the ant need to touch?
- How many full circles does the ant need to make?
- How many points in all has the ant touched in the 5 trips around?
- How many more letters must the ant touch?
- What are the next three letters?
- What is the last point the ant will touch?

Notes for next time:

The KEY to Solving Mathematical Problems:
Stop! Think…Go…Think…Go…Think…Go…

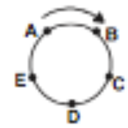
Problem #9: An ant travels around the circle in the direction shown. It touches each of the labeled points in order. The first three points that the ant touches are A, B, and C, in that order. What is the 28th point that the ant touches?



Answer

Solution Strategy Worksheet for Problem #9

Problem #9: An ant travels around the circle in the direction shown. It touches each of the labeled points in order. The first three points that the ant touches are A, B, and C, in that order. What is the 28th point that the ant touches?

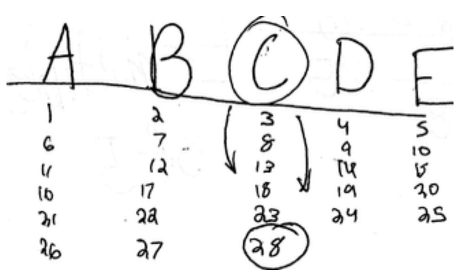


STOP! Write an open sentence:

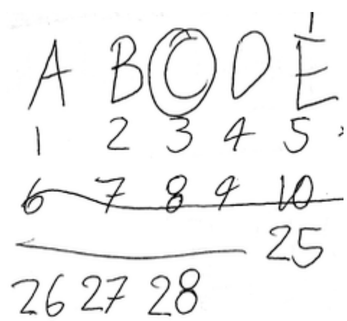
Think…Go…Think…Go…Think…Go…

1. Mathletes A and B used similar strategies to obtain a correct solution.
    a. What did Mathlete A do?
    b. What did Mathlete B do?
    c. Which strategy is more efficient? Why?
2. Mathletes C and D also used similar strategies, which one is incorrect solution?
    a. What did Mathlete C do? Is the answer correct? If not, why not?
    b. What did Mathlete D do? Is the answer correct? If not, why not?
    c. For the incorrect answer, where was the error in thinking?

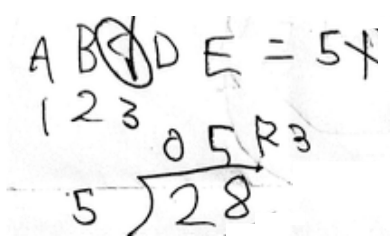
A:



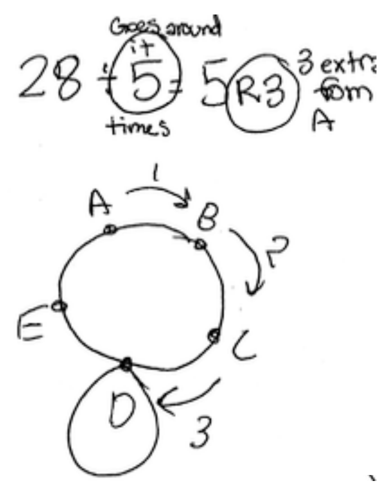
B:



C:



D:

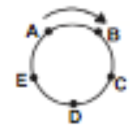


Now REVIEW YOUR WORK:

- If you solved the problem correctly, can you improve your strategy to be more efficient?
- If you did not get the correct answer, can you find your error and correct your work?

Answers to Solution Strategy Worksheet for Problem #9

Problem #9: An ant travels around the circle in the direction shown. It touches each of the labeled points in order. The first three points that the ant touches are A, B, and C, in that order. What is the 28th point that the ant touches?



STOP! Write an open sentence:
[28th point = __.]

Think…Go…Think…Go…Think…Go…

1. Mathletes A and B used similar strategies to obtain a correct solution.
   a. What did Mathlete A do?
   [A made a table with all the points in the top row and all the numbers from 1 to 28 below.]
   b. What did Mathlete B do?
   [B made a table with all the points in the top row, counted by five to 25, and then listed the last three numbers.]
   c. Why is strategy B more efficient than strategy A?
   [Counting by fives is faster than counting by ones.]
2. Mathletes C and D also used similar strategies. Which one is an incorrect solution?
   a. What did Mathlete C do?
   [Mathlete C divided 5 into 28 to get 5, remainder 3, and then listed the points A, B, C, D, E and counted the remainder of 3 points, starting with A.]

   Is the answer correct? *[Yes.]*
   b. What did Mathlete D do?
   [Mathlete D divided 5 into 28 to get 5, remainder 3 and then counted steps away from A, without counting point A itself.]

   Is the answer correct? *[No.]*
   c. For the incorrect answer, where was the error in thinking? *[Mathlete G reasoned that after touching all five points five times, the ant would end on point A, but the ant would actually be on E at that moment. The ant doesn't reach A again until the beginning of the 6th time around.]*